Trait dimensionality effects on model communities

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Joint work with Z.Miller, J.A. Capitán, T.Bodnar and S. Allesina

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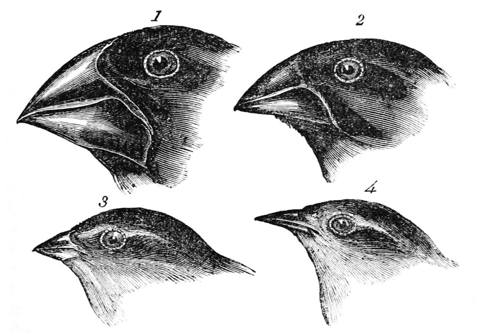
Functional traits



1. Geospiza magnirostris. 2. Geospiza fortis.
3. Geospiza parvula. 4. Certhidea olivasea.

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In our case $\mathcal{T} = \mathbb{R}^k$ and:

$$\begin{aligned} g &: \mathcal{T} \to \mathbb{R}, t \to 1 \\ f &: \mathcal{T} \times \mathcal{T} \to \mathbb{R}, (t, t') \to \langle t, t' \rangle \end{aligned}$$

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For $x \in \mathbb{R}^n$:

$$\frac{dx}{dt} = x \circ (r - Ax)$$

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$$r = 1$$

$$A_{ij} = \frac{1}{k} f(t_i, t_j)$$

$$G = [t_{ji}] \in \mathbb{R}^{k \times n}$$

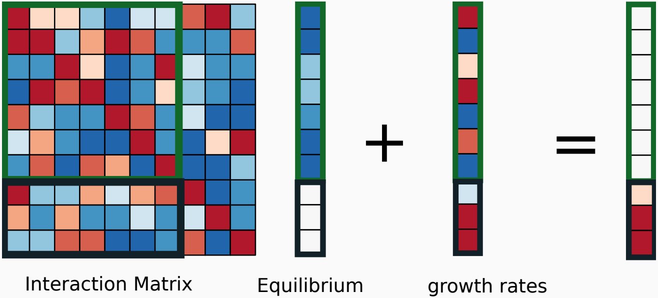
$$A = \frac{1}{k} G^T G$$

Attractors

(**Thm 15.3.1 Hofabuer and Sigmund 1998**) Since A is symmetric and positive definite ($k \geq n$) we have a unique globally stable fixed point parameterized by $S \subset N = \{1, \ldots, n\}$:

$$\begin{aligned} x_i > 0 \ , \ i \in S & \quad \text{(Feasibility)} \\ x_S(Ax_S + r) = 0 & \quad \text{(Equilibrium)} \\ (Ax_S + r)_i < 0 \ , \ i \notin S & \quad \text{(Non-invasibility)} \end{aligned}$$

Non-invasible/saturated fixed point



Random zoo

- Jose A. Capitán (Universidad Politécnica de Madrid)
- Jacopo Grilli (ICTP)
- Kent E. Morrison (American Institute of Mathematics)
- Stefano Allesina (Chicago)

Random zoo

(Top Down approach)

- Take a pool of n species.
- Let dynamics elapse.
- $k \leq n$ species are coexisting.
- We want to determine $P(k|n)$.
- Study random ecosystems.

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(For symmetric stable systems, equivalent to **Bottom up approach**)

Random Zoo

Computed the distribution of sizes of the survival community for the cases: (A_{ij}) and r_i *symmetric* about 0 random variables

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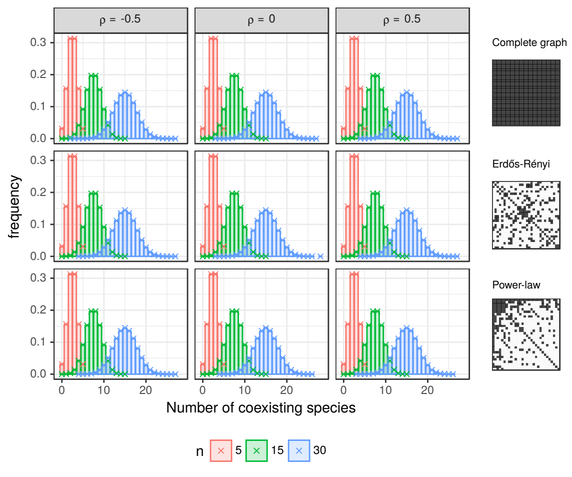
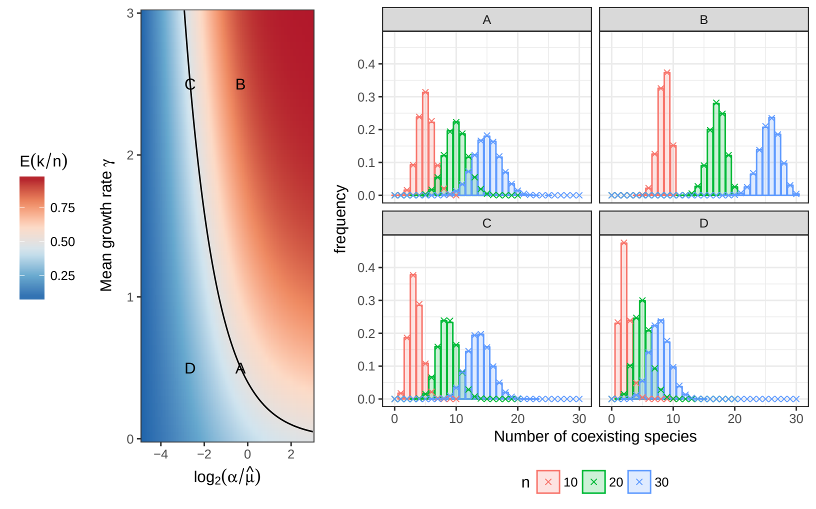


Nonzero mean

(A_{ij}) deterministic and constant off diagonal with r gaussian.



Sample Covariance matrix

Necessary condition for non-degenerate equilibrium $k \geq n$.

$$G_i \sim \mathcal{N}(0, \Sigma) \qquad \textbf{(Gaussian distribution)}$$

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$$A \sim \mathcal{W}_n(\frac{1}{k}\Sigma, k) \quad \textbf{(Wishart Distribution)}$$

Observables

How does the community "look" as a function of the number of traits k?

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- Distribution of the number of survivals : $\mathbb{P}(|S||k, n, \Sigma)$
- Mean number of survivals : $\wp(k, n, \Sigma)$.
- Total biomass at the attractors : $W(k, n, \Sigma)$.

Attractors

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Probability of feasibility

$$\begin{aligned} Ax &= 1 \\ x_i &> 0 \end{aligned}$$

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Let $A \sim \mathcal{W}_n(\Sigma, k)$, $1_n \in \mathbb{R}^n$ a vector of ones and $L_{n-1} = (I_{n-1} 0)$, then (Proof of Thm 1. Bodnar and Okhrin 2011)

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$$\tilde{x} = \frac{L_{n-1} A^{-1} 1_n}{1_n^T A^{-1} 1_n}$$

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$$\tilde{x} = \frac{L_{n-1} A^{-1} 1_n}{1_n^T A^{-1} 1_n}$$
$$\tilde{x} \sim t_{n-1}\left(k - n + 2; \frac{L_{n-1} \Sigma^{-1} 1_n}{1_n^T \Sigma^{-1} 1_n}, \frac{1}{(k-n+2) 1_n^T \Sigma^{-1} 1_n} L_{n-1} R_1 L_{n-1}^T\right)$$

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$$R_1 = \Sigma^{-1} - \Sigma^{-1} 1_n 1_n^T \Sigma^{-1} / 1_n^T \Sigma^{-1} 1_n$$

Probability of feasibility

$$P_f(n) = \int_{\mathbb{R}^{n-1}} d\tilde{x}^{n-1} p(\tilde{x}) \Theta(1 - 1_{n-1}^T \tilde{x}) \prod_i \Theta(\tilde{x}_i)$$

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$$P_f(n) = \int_{\mathbb{R}} du g(u) \mathbb{P}(y_u \succ 0, 1_{n-1}^T y_u < 1) \tag{1}$$

Where:

$$\begin{aligned} u &\sim \chi^2_{k-n+2} \\ y_u &\sim \mathcal{N}\left(\frac{L_{n-1}\Sigma^{-1}1_n}{1_n^T \Sigma^{-1} 1_n}, \frac{1}{u 1_n^T \Sigma^{-1} 1_n} L_{n-1} R_1 L_{n-1}^T\right) \end{aligned} \tag{2}$$

Probability of Non-Invasibility

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

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(Thm 3.2.10 Muirhead 1982):

$$A_{21}|A_{11} \sim \mathcal{N}(\Sigma_{21}\Sigma_{11}^{-1}A_{11}, \Sigma_{22.1} \otimes A_{11})$$
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In particular:

$$A_{21}A_{11}^{-1}1|A_{11} \sim \mathcal{N}(\Sigma_{21}\Sigma_{11}^{-1}1, 1^T A_{11}^{-1}1\Sigma_{22.1})$$

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$$P_{ni}(m) = \int_{\mathbb{R}_+} dwg(w)O^-(n-m, 1-\Sigma_{21}\Sigma_{11}^{-1}1, w\Sigma_{22.1})$$

Probability of Non-Invasibility

Theorem 3.2.11 Muirhead 1982 implies that :

$$\frac{1^T \Sigma_{11}^{-1} 1}{W} \sim \chi^2_{k-m+1}$$

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$$P_{ni}(m) = \int_{\mathbb{R}_+} dw f(w) O^-(n-m, 1 - \Sigma_{21}\Sigma_{11}^{-1}1, \frac{1^T \Sigma_{11}^{-1} 1}{w} \Sigma_{22.1}) \qquad (3)$$

For f the density function of χ^2_{k-m+1}.

Constant Correlation $\rho \geq 0$ - Feasibility

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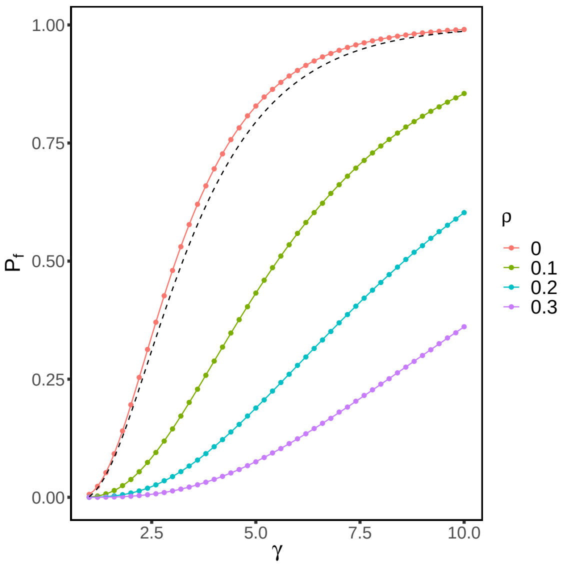
$$P_f(n) = \int_{\mathbb{R}_+} du g(u) \frac{-i\sqrt{n\alpha_u}}{\sqrt{2\pi}} \int_\Gamma d\zeta e^{\frac{n\zeta^2\alpha_u}{2}} \Phi(\frac{1/n + \zeta\alpha_u}{\sqrt{\alpha_u}})^n \tag{4}$$

$$\alpha_u := \frac{1+(n-1)\rho}{un(1-\rho)}$$

$$\beta_u := \frac{\alpha_u}{n}$$

Constant Correlation $\rho \geq 0$ - Feasibility



Constant Correlation $\rho \geq 0$ - Invasibility

$$P_{ni}(m,n) = \int_{\mathbb{R}^+} dw f(w) \int_{\mathbb{R}} dy \phi(y) \Phi\left(\frac{-1/m + y\sqrt{\beta_w}}{\sqrt{\alpha_w}}\right)^{n-m} \tag{5}$$

$$\alpha_w = \frac{1+(m-1)\rho}{mw(1-\rho)}$$

$$\beta_w = \frac{\rho\alpha_w}{1+(m-1)\rho}$$

Distribution of survivors

$$\mathbb{P}(|S| = m|n, k, \rho) = \binom{n}{m} P_{f,n_i}(m) = \binom{n}{m} P_f(m) P_{ni}(m) = P_a(m) \quad (6)$$

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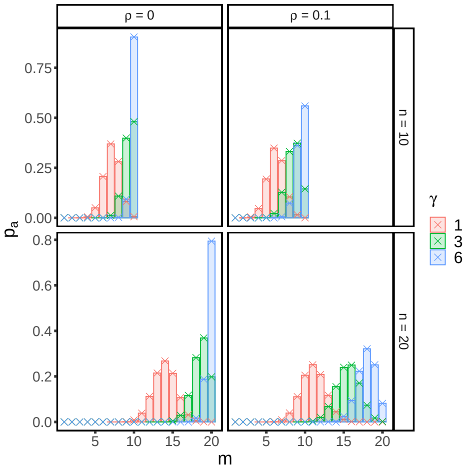

Approximations for $\wp$

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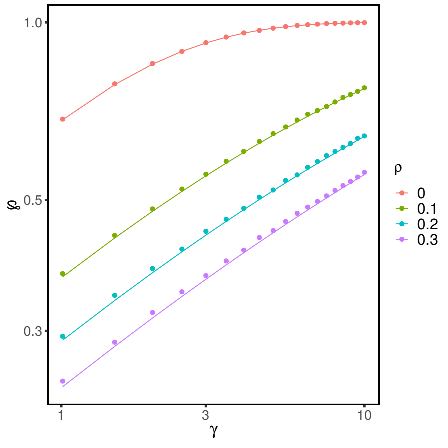
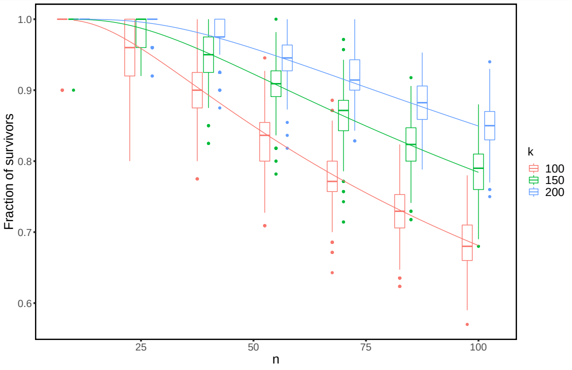


Signature of Trait space dimension



Phylogenetic correlations

Let T be a rooted phylogenetic tree, with total time 1.

$$\Sigma_T(i,j) = 1 - d(i,j) \tag{8}$$

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In the limit of $\gamma \to \infty$, then $\frac{1}{k}A \to \Sigma_T$.

Properties of Phylogenetic covariances

- Any subset of species coexist (recursive proof)

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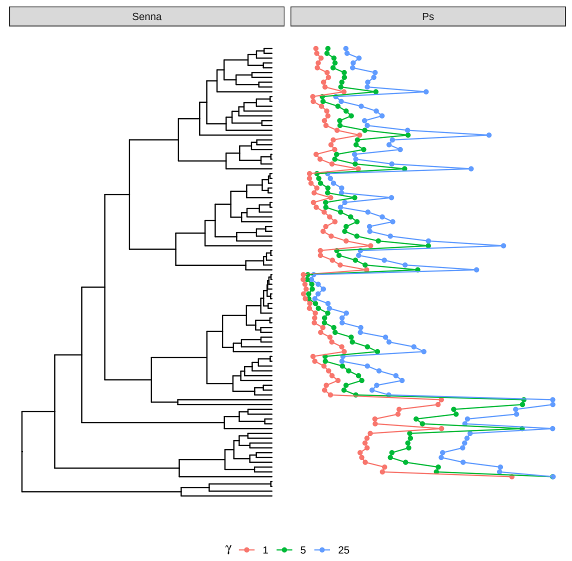
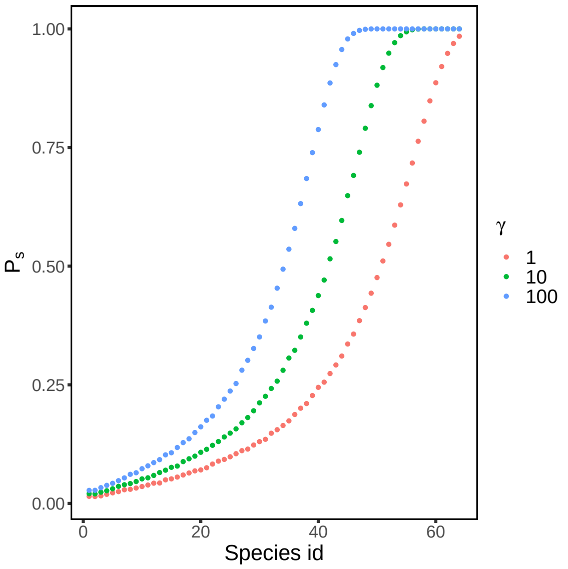
What happens in the case of γ *finite*?

Species sorting

(Weber and Agrawal 2014)



Perfectly unbalanced tree



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- We can detect a signature of the dimension of the trait space (under independent trait values).
- Our framework reproduces species sorting under phylogenetic correlation.

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- Stefano Allesina
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