Tutorial on Random Matrix Theory

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Quick reminder of the motivations

The paradigm of May ($\sim$ 1971)

General idea : for complex multispecies networks, detailed interactions are rarely available, random matrix models may be relevant.

More precisely, an ecological network is modelled as a system of ODEs of the form:

$$\frac{da_i(t)}{dt} = f_i(a_1, \ldots, a_N), \; 1 \leq i \leq N,$$

with a_i the abundance of species i.

Assume the existence of an equilibrium $(a_1^*, \ldots, a_N^*)$ and linearise near equilibrium. Namely, if we set $a_i(t) = a_i^* + x_i(t)$, the dynamics can be approximated at first order by the linear system:

$$\frac{d}{dt}\mathbf{x}(t) = \mathcal{J}(\boldsymbol{a^*}) \cdot \mathbf{x}(t),$$

with $\mathcal{J}(\boldsymbol{a^*})_{k\ell} := \frac{\partial f_k}{\partial x_\ell}(a_1^*, \ldots, a_N^*)$.

May's model : the entries of $\mathcal{J}$ are i.i.d. random variables (except on the diagonal).

Lotka-Volterra models

The dynamics of interacting species may be described by the Lotka-Volterra equations:

$$\frac{da_i(t)}{dt} = a_i \left(r_i - \theta a_i + \sum_{j=1}^{N} \frac{Z_{ij}}{N^\delta} a_j \right)$$

where

- r_i is the intrinsic growth rate of species i
- $\theta > 0$ is a friction coefficient (intraspecific competition)
- Z_{ij} stands for the interactions $j \to i$.

The equilibrium $\boldsymbol{a}^*$ is given by

$$\boldsymbol{a}^* = \left(\theta I_N - \frac{Z}{N^\delta} \right)^{-1} \boldsymbol{r}$$

The Jacobian $\mathcal{J}(\boldsymbol{a}^*)$, which is explicit

$$\mathcal{J}(\boldsymbol{a}^*) = \mathrm{diag}(\boldsymbol{a}^*) \left(-\theta I_N + \frac{Z}{N^\delta} \right)$$

It is a special case of *density-dependent models*, for which

$$\mathcal{J}(\boldsymbol{a}^*) = \mathrm{diag}(\boldsymbol{a}^*) \cdot \boldsymbol{M},$$

where $\boldsymbol{M}$ may depend on $\mathrm{diag}(\boldsymbol{a}^*)$.

Large Random Matrices

Random matrices

It is a $N \times N$ matrix

$$\mathbf{Y}_N = \begin{bmatrix} Y_{11} & \cdots & Y_{1N} \\ \vdots & & \vdots \\ Y_{N1} & \cdots & Y_{NN} \end{bmatrix}$$

whose entries $(Y_{ij};\ 1 \le i, j \le N)$ are random variables.

Matrix features

Of interest are the following quantities

- $\mathbf{Y}_N$'s **spectrum** $(\lambda_i,\, 1 \le i \le N)$ and eigenvectors
- **Extreme eigenvalues** ($\lambda_{\min}$ and $\lambda_{\max}$ if spectrum is real, spectral radius, eigenvalue with maximal real part etc.)
- **Linear statistics** $\mathrm{Trace}\, f(\mathbf{Y}_N) = \sum_{i=1}^N f(\lambda_i)$ and their fluctuations

Asymptotic regime

Often, the description of the previous features **takes a simplified form** as

$$\boxed{N \to \infty}$$

Wigner Matrices

Matrix model

Let $\mathbf{X}_N = (X_{ij})$ a real symmetric (or Hermitian) $N \times N$ matrix with i.i.d. entries **on and above** the diagonal with

$$\mathbb{E}X_{ij} = 0 \text{ and } \mathbb{E}\left|X_{ij}\right|^2 = 1$$

and $X_{ij} = \overline{X_{ji}}$ (for symmetry).

- consider the spectrum of **Wigner matrix** $\boxed{\mathbf{Y}_N = \frac{\mathbf{X}_N}{\sqrt{N}}}$

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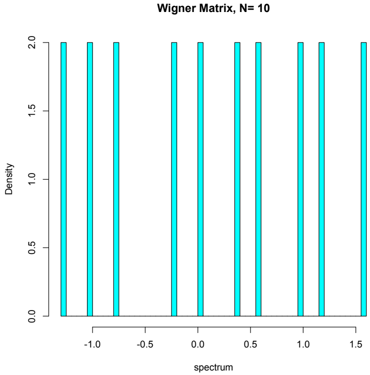

Figure: Histogram of the eigenvalues of $\mathbf{Y}_N$

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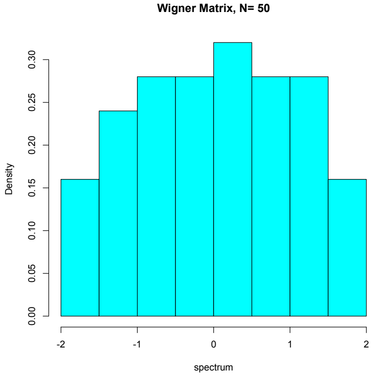

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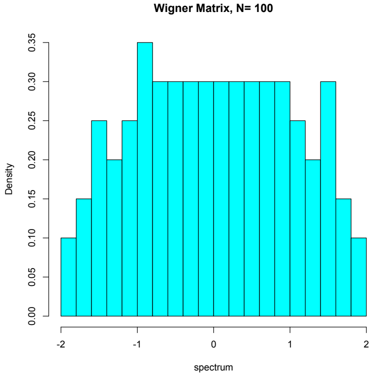

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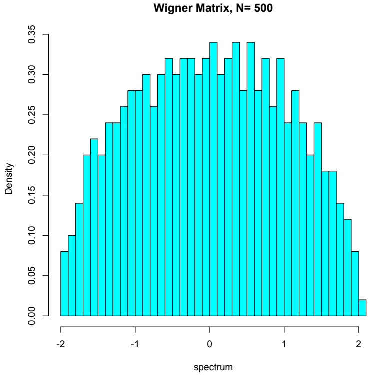

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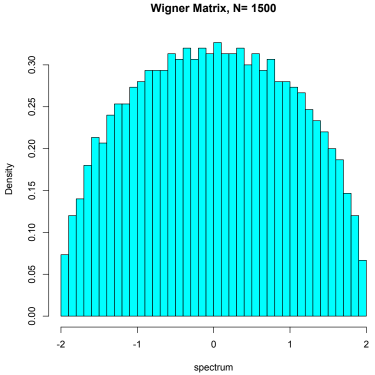

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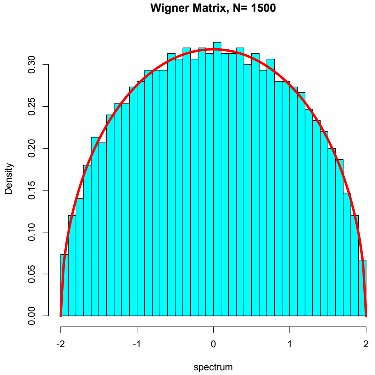


Figure: The semi-circular distribution (in red) with density $x \mapsto \frac{\sqrt{4-x^2}}{2\pi}$

Wigner's theorem (1948)

"The histogram of a Wigner matrix converges to the **semi-circular distribution**"

About normalisation

Wigner's theorem can be written as follows :

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i(\mathbf{Y}_N)) \xrightarrow[N\to\infty]{a.s.} \int_{-2}^{2} f(x)\frac{\sqrt{4-x^2}}{2\pi}\mathrm{d}x = \int f(x)\mathrm{d}\sigma(x).$$

A quick computation

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\lambda_i(\mathbf{Y}_N)^2 &= \frac{1}{N}\mathrm{Trace}(\mathbf{Y}_N)^2 = \frac{1}{N^2}\mathrm{Trace}(\mathbf{X}_N)^2 \\ &= \frac{1}{N^2}\sum_{i,j=1}^{N} X_{ij}X_{ji} = \frac{1}{N^2}\sum_{i,j=1}^{N}|X_{ij}|^2\end{aligned}$$

so that

$$\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\lambda_i(\mathbf{Y}_N)^2\right) = 1.$$

In the Lotka-Volterra model, this normalisation corresponds to $\delta = 1/2$.

Fluctuations of linear statistics

Many results starting with Johansson (1999) lead to the following results: if the entries have enough finite moments,

- for smooth test functions f,

$$N\left(\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i(\mathbf{Y}_N)) - \int f(x)\mathrm{d}\sigma(x)\right)$$

  converges to a Gaussian variable, with mean and variance depending on f.

- for $f(x) = \mathbf{1}_{[y,\infty)}(x)$, we let $N_y(\mathbf{Y}_N) := \sharp\{i; \lambda_i(\mathbf{Y}_N)) \geq y\}$. If the first fourth moment match with a normal variable, $\mathrm{Var}(N_y(\mathbf{Y}_N)) = \left(\frac{1}{2\pi^2} + o(1)\right)\log N$, and

$$\frac{N_y(\mathbf{Y}_N) - N\sigma([y,\infty))}{\sqrt{\mathrm{Var}(N_y(\mathbf{Y}_N))}}$$

  converges to a normal random variable.

Local spectrum behavior for Wigner matrices

Let

$$\lambda_{\max} = \lambda_{\max}(\mathbf{Y}_N) \qquad \text{and} \qquad \lambda_{\min} = \lambda_{\min}(\mathbf{Y}_N)$$

Theorem (convergence of extremal eigenvalues)

If $\mathbb{E}|X_{ij}|^4 < \infty$, then

$$\lambda_{\max} \xrightarrow[N\to\infty]{\text{a.s.}} 2 \qquad \text{and} \qquad \lambda_{\min} \xrightarrow[N\to\infty]{\text{a.s.}} -2$$

Theorem (fluctuations of $\lambda_{\max}$ and Tracy-Widom distribution)

If $\lim_{s\to\infty} s^4\mathbb{P}(|X_{12}| \geq s) = 0$, we have

$$N^{2/3}\left\{\lambda_{\max}(\mathbf{Y}_N) - 2\right\} \xrightarrow[N\to\infty]{\mathcal{L}} \mathbb{P}_{\mathrm{TW}}.$$

Remark

- Notice the **strange normalization** $N^{2/3}$
- This stems from **the $\sqrt{x}$-shape of the density at the edge** (see later)

Details on Tracy-Widom distribution

Tracy-Widom distribution is defined by

- its cumulative distribution function

$$F_{TW}(x) = \exp\left\{ -\int_x^{\infty} (u-x)^2 q^2(u)\, du \right\}$$

- where

$$q''(x) = xq(x) + 2q^3(x) \quad \text{and} \quad q(x) \sim \mathrm{Ai}(x) \text{ as } x \to \infty \ .$$

  $x \mapsto \mathrm{Ai}(x)$ being the Airy function.

Don't bother .. just download it

- For simulations, cf. `R Package 'RMTstat'`, by Johnstone et al.
- Also, Folkmar Bornemann (TU München) has developed fast `matlab` code

Tracy-Widom curve

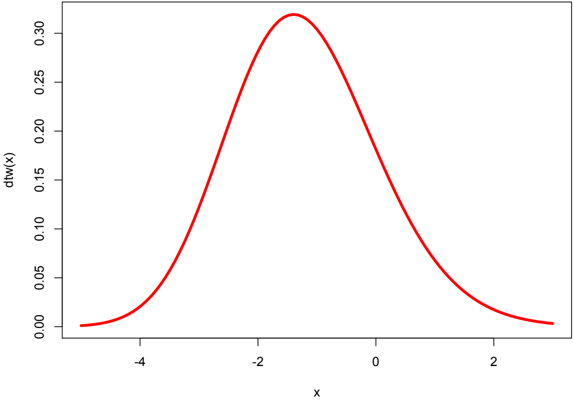


Figure: Tracy-Widom density

A heuristic on the normalization $N^{2/3}$

- By Wigner's theorem, $\boxed{\frac{\#\{\lambda_i > 2-\varepsilon\}}{N} \longrightarrow \int_{2-\varepsilon}^{2} \frac{\sqrt{4-x^2}}{2\pi}\,\mathrm{d}x}$
- Hence "for small ε",

$$\begin{aligned} \#\{\lambda_i > 2-\varepsilon\} &\approx N\int_{2-\varepsilon}^{2} \frac{\sqrt{4-x^2}}{2\pi}\,\mathrm{d}x \\ &\approx N\frac{4}{2\pi}\int_{2-\varepsilon}^{2}\sqrt{2-x}\,dx = cN\varepsilon^{3/2} \end{aligned}$$

- To have **finitely many values** in $(2-\varepsilon,\infty)$, we want $\boxed{\#\{\lambda_i > 2-\varepsilon\} = \mathcal{O}(1)}$
- We choose $\varepsilon = cN^{-2/3}$ so that $\boxed{N\varepsilon^{3/2} = \mathcal{O}(1)}$ and

$$\#\{\lambda_i > 2 - cN^{-2/3}\} = \#\{N^{2/3}(\lambda_i - 2) > c\} = \mathcal{O}(1)$$

- This suggests to study the fluctuations of

$$\boxed{N^{2/3}\,(\lambda_{\max} - 2)}$$

- The $N^{2/3}$ normalization is strongly associated to the $\sqrt{x}$-behaviour of the density at the corresponding edge

Marčenko-Pastur's theorem

Theorem

- Consider a $N \times n$ matrix $\mathbf{X}_N$ with i.i.d. entries

$$\mathbb{E}X_{ij} = 0 \ , \quad \mathbb{E}|X_{ij}|^2 = \sigma^2 \ .$$

with N and n **of the same order** and L_N the **spectral measure** of $\frac{1}{n}\mathbf{X}_N\mathbf{X}_N^*$:

$$c_n \triangleq \frac{N}{n} \xrightarrow[n\to\infty]{} c \in (0,\infty) \ , \quad L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(n^{-1}\mathbf{X}_N\mathbf{X}_N^*)} \ .$$

- Then **almost surely** (= for almost every realization)

$$L_N \xrightarrow[N,n\to\infty]{} \mathbb{P}_{\check{\mathrm{M}}\mathrm{P}} \quad \text{in distribution}$$

where $\mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}$ is **Marčenko-Pastur** distribution:

$$\boxed{\mathbb{P}_{\check{\mathrm{M}}\mathrm{P}}(dx) = \left(1 - \frac{1}{c}\right)_+ \delta_0(dx) + \frac{\sqrt{[(\lambda^+ - x)(x - \lambda^-)]_+}}{2\pi\sigma^2 xc}\, dx}$$

$$\text{with} \quad \left\{ \begin{array}{lcl} \lambda^- & = & \sigma^2(1-\sqrt{c})^2 \\ \lambda^+ & = & \sigma^2(1+\sqrt{c})^2 \end{array} \right.$$

Simulations vs MP distribution

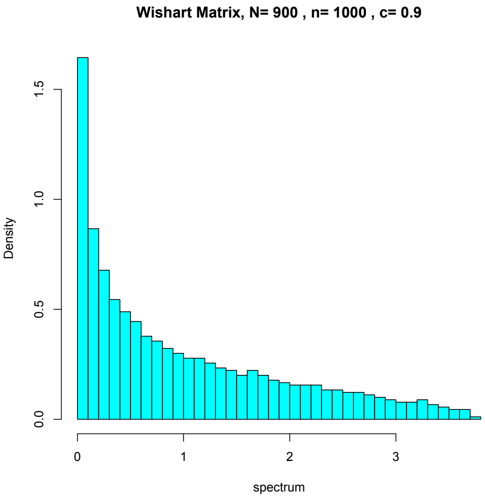


Figure: Histogram of $\frac{1}{n}\mathbf{X}_N\mathbf{X}_N^*$, $\sigma^2 = 1$

Simulations vs M̌P distribution

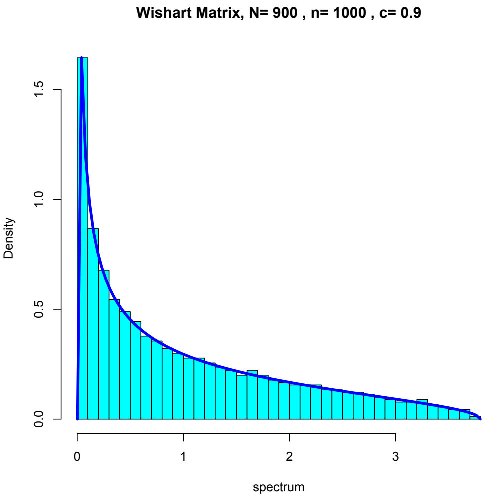
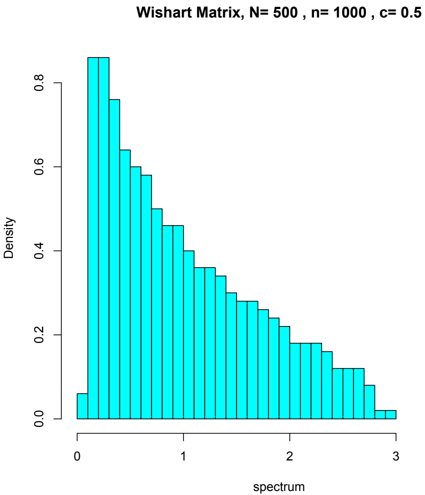


Figure: Marčenko-Pastur distribution for $c = 0.9$

Simulations vs MP distribution



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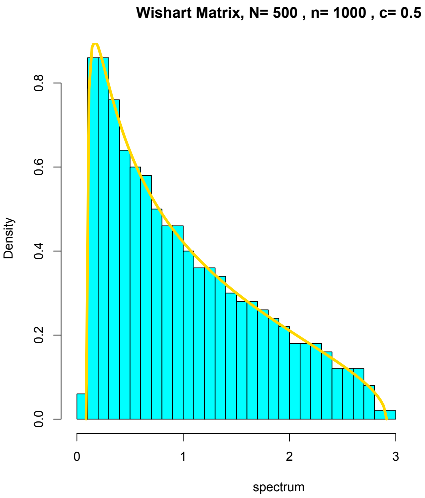

Figure: Marčenko-Pastur distribution for $c = 0.5$

Simulations vs M̌P distribution

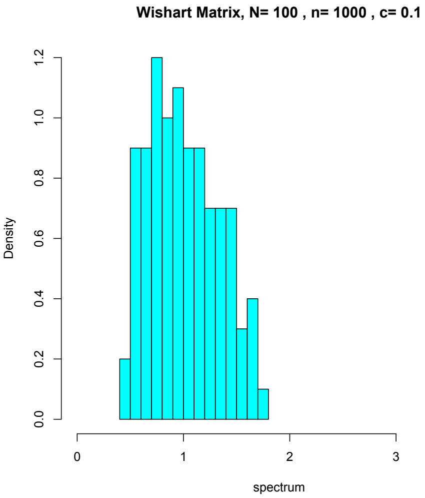

Figure: Histogram of $\frac{1}{n}\mathbf{X}_N\mathbf{X}_N^*$, $\sigma^2 = 1$

Simulations vs M̌P distribution

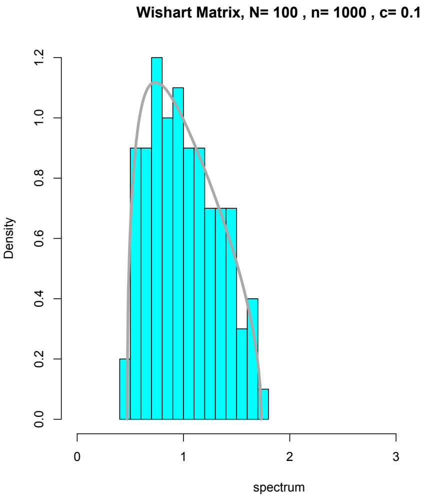

Figure: Marčenko-Pastur distribution for $c = 0.1$

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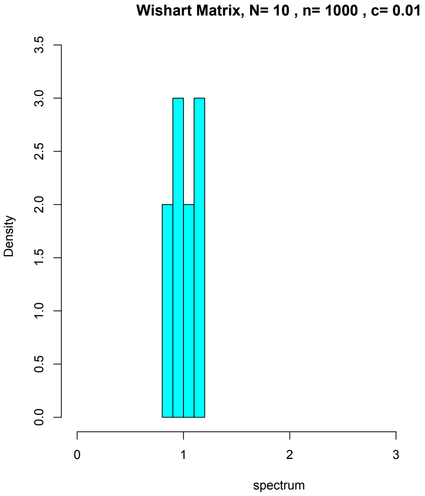

Figure: Histogram of $\frac{1}{n}\mathbf{X}_N\mathbf{X}_N^*$, $\sigma^2 = 1$

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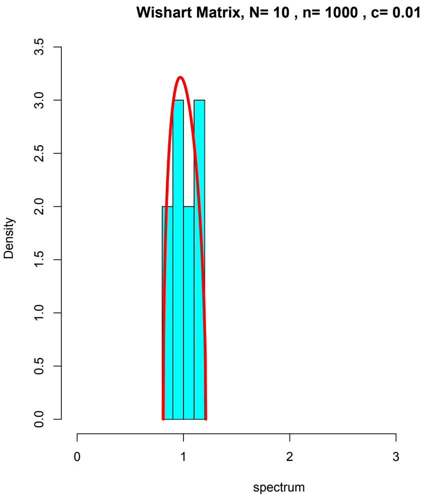

Figure: Marčenko-Pastur distribution for $c = 0.01$

Theorem (fluctuations of $\lambda_{\max}$ and Tracy-Widom distribution)

$$\frac{N^{2/3}}{\Theta_N} \left\{ \lambda_{\max}\left(\frac{1}{n}\mathbf{X}_N \mathbf{X}_N^*\right) - \sigma^2(1+\sqrt{c_n})^2 \right\} \xrightarrow[N,n\to\infty]{\mathcal{L}} \mathbb{P}_{\mathrm{TW}}$$

where

$$c_n = \frac{N}{n} \quad \text{and} \quad \Theta_N = \sigma^2(1+\sqrt{c_n})\left(\frac{1}{\sqrt{c_n}}+1\right)^{1/3}$$

Remark

In this case, we also have Gaussian fluctuations of the linear statistics in the same scale as for the other models.

Non-Hermitian Matrices and the circular law

Matrix model

Let $\mathbf{X}_N$ be a $N \times N$ matrix

$$\mathbf{X}_N = \begin{pmatrix} X_{11} & \cdots & X_{1N} \\ \vdots & & \vdots \\ X_{N1} & \cdots & X_{NN} \end{pmatrix}$$

- Consider matrix $\boxed{\mathbf{Y}_N = \frac{1}{\sqrt{N}} \mathbf{X}_N}$
- Beware that the eigenvalues are **complex**!

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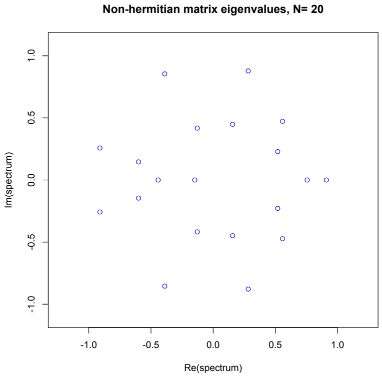

Figure: Distribution of $\mathbf{Y}_N$'s eigenvalues

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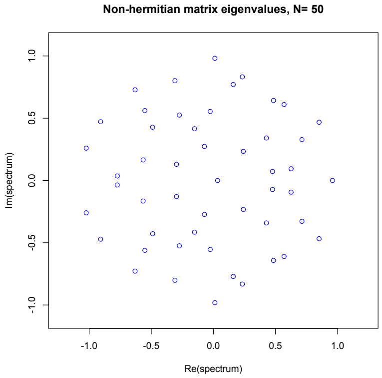

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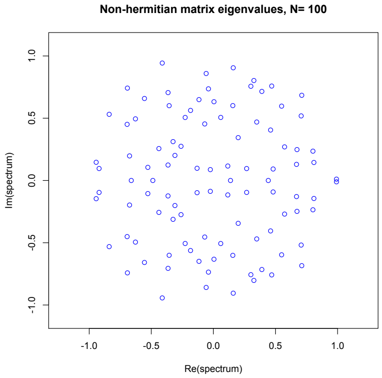

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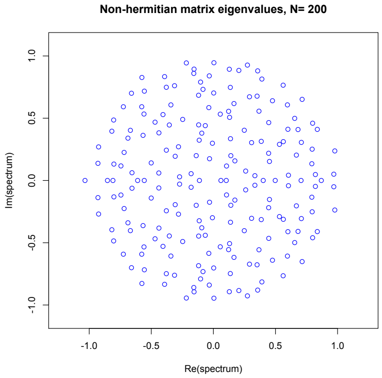

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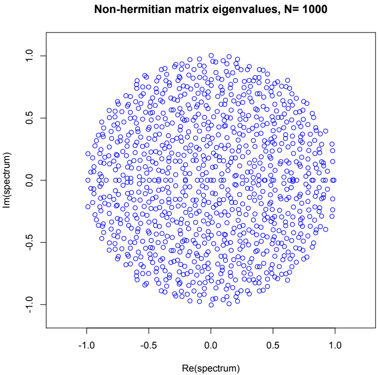

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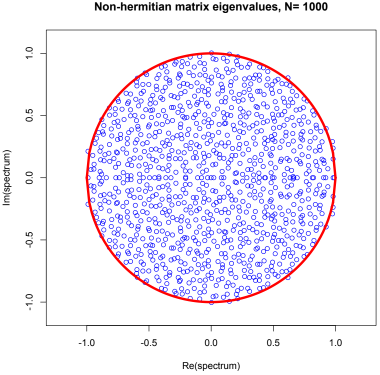

Figure: The circular law (in red)

Theorem: The Circular Law (Ginibre, Mehta, Girko, Tao & Vu, etc.)

The spectrum of $\mathbf{Y}_N$ converges to **the uniform probability on the disc**

The circular law

Theorem

Assume that $\mathbb{E}(X_{ij}) = 0$ and $\mathbb{E}|X_{ij}|^2 = \sigma^2$. Let $Y_N = \frac{X_N}{\sqrt{N}}$, then

$$\boxed{\text{a.s.} \qquad \mu_N^Y = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i(Y_N)} \quad \xrightarrow[N\to\infty]{\text{weakly}} \quad \frac{1}{\pi\sigma^2}\mathbf{1}_{\{x^2+y^2\le\sigma^2\}}\,dx\,dy}$$

The spectral radius

Assume $\mathbb{E}|X_{ij}|^4 < \infty$ and consider the spectral radius

$$\rho(Y_N) = \max\{|\lambda|\,,\ \lambda \in \text{spectrum}(Y_N)\}$$

Then

$$\boxed{\rho(Y_N) \quad \xrightarrow[N\to\infty]{a.s.} \quad \sigma}$$

Link for May's criteria for stability

$\mathcal{J} = -I_n + \mathbf{X_N}$, with X_{ij} is $\mathrm{Ber(C)} \cdot \mathcal{N}(0, \sigma^2)$, so that the equilibrium is stable iif

$$\boxed{\sqrt{NC}\sigma < 1.}$$

Fluctuations in the non-Hermitian case

Fluctuations of linear statistics (Rider, Silverstein)

Under appropriate assumptions on the moments of X_{12}, for any analytic function f,

$$N\left(\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i(\mathbf{Y}_N)) - f(0)\right)$$

converges to a Gaussian variable, with mean zero and variance $\frac{1}{\pi}\int |f'(z)|^2 \mathrm{d}z$.

Fluctuations of spectral radius (Kostlan)

If X_{11} is a Gaussian random variable (Ginibre matrix), then $\rho(\mathbf{Y}_N)$ has Gumbel fluctuations. More precisely, if $c_N := \log N - 2\log\log N - \log(2\pi)$, for any $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\left(2\sqrt{Nc_N}\left(\rho(\mathbf{Y}_N) - \left(1 + \frac{1}{2}\sqrt{\frac{c_N}{N}}\right)\right) \leq x\right) = e^{-e^{-x}}.$$

Another model of non-Hermitian matrices : elliptic law

Model

- the diagonal entries are iid and independent of the rest of the matrix, $EX_{11} = 0$ and $EX_{11}^2 < \infty$,
- for $i \neq j$, the pair (X_{ij}, X_{ji}) is independent of the rest of the matrix, iid

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- $EX_{12} = EX_{21} = 0$, $EX_{12}^2 = EX_{21}^2 = 1$, $E|X_{12}|^4 < \infty, E|X_{21}|^4 < \infty$, $E(X_{12}X_{21}) = \rho \in (-1, 1)$.

As before, we set $\boxed{\mathbf{Y}_N = \frac{1}{\sqrt{N}}\mathbf{X}_N}$

Global behavior (Fyodorov, Girko, Khoruzhenko, Naumov, etc.)

The empirical distribution of the eigenvalues converges to the uniform distribution on the ellipse :

$$\boxed{\mathcal{E}_\rho := \left\{(x, y) \in \mathbb{R}^2; \frac{x^2}{(1+\rho)^2} + \frac{y^2}{(1-\rho)^2} \leq 1\right\}}$$

Local behavior (Renfrew)

For any $\delta > 0$, a.s. for N large enough, there is no eigenvalue of $\mathbf{Y}_N$ outside a δ-neighborhood of $\mathcal{E}_\rho$.

Spiked models

General idea

If you take your favorite random matrix model and add a deterministic matrix with low rank (that is the rank stays bounded as N goes to infinity), then

- the global behavior is not affected
- the behavior of extreme eigenvalues can be affected drastically if the perturbation is strong enough.

There is a huge litterature on the subject...
The simplest case can be stated as follows :

Wigner case

We denote by $\mathbf{\Theta}_N$ a rank one matrix with eigenvalue θ and delocalized eigenvectors. For example,

$$\mathbf{\Theta}_N := \theta \mathbf{u}\mathbf{u}^*,$$

with u uniform on the sphere in dimension N. Let $\mathbf{Y}_N$ be a Wigner matrix as defined earlier and $\lambda_{\max} := \lambda_{\max}(\mathbf{Y}_N + \mathbf{\Theta}_N)$.
Then,

- if $\theta \leq 1$, $N^{2/3}\left(\lambda_{\max} - 2\right)$ converges to a Tracy-Widom distribution,
- if $\theta > 1$, $\sqrt{N}\left(\lambda_{\max} - \left(\theta + \frac{1}{\theta}\right)\right)$ converges to a Gaussian distribution.

Simulations

Simulations

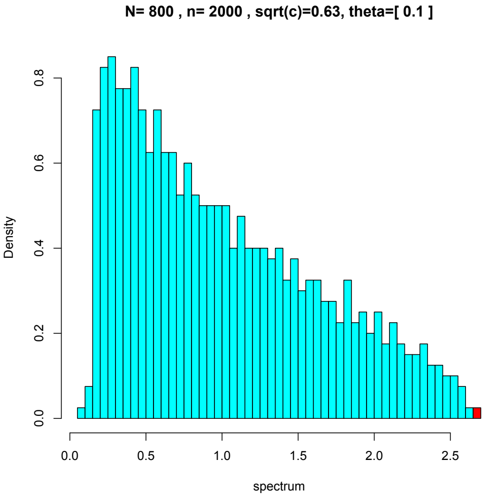

Figure: Spiked model - strength of the perturbation $\theta = 0.1$

Simulations

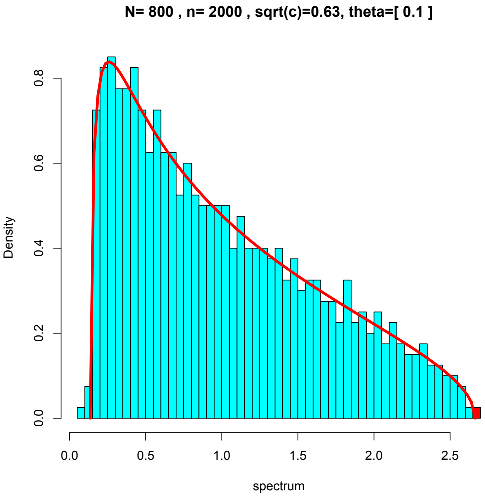

Figure: Spiked model - strength of the perturbation $\theta = 0.1$

Simulations

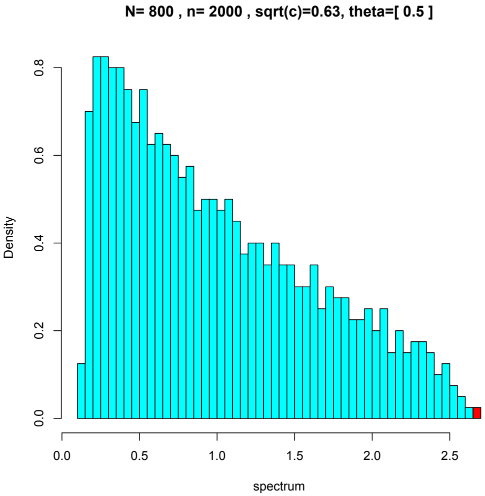

Figure: Spiked model - strength of the perturbation $\theta = 0.5$

Simulations

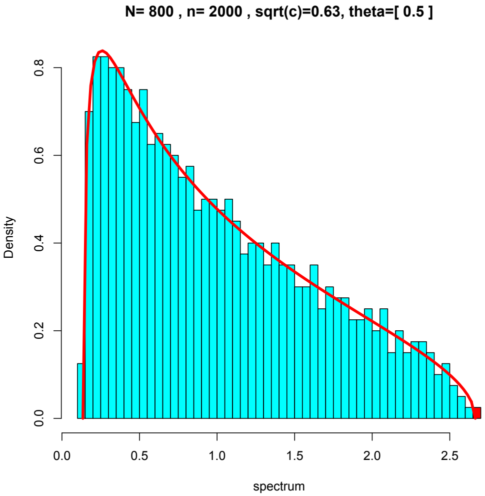

Figure: Spiked model - strength of the perturbation $\theta = 0.5$

Simulations

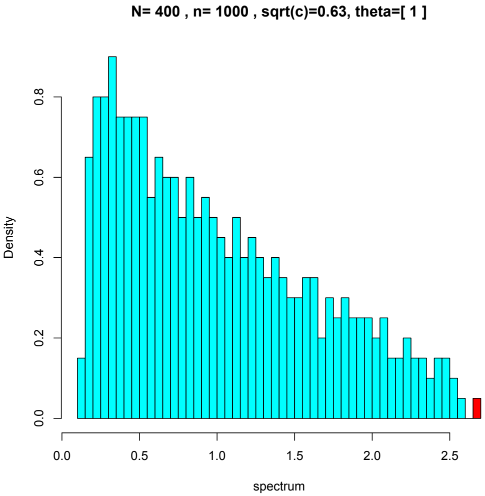

Figure: Spiked model - strength of the perturbation $\theta = 1$

Simulations

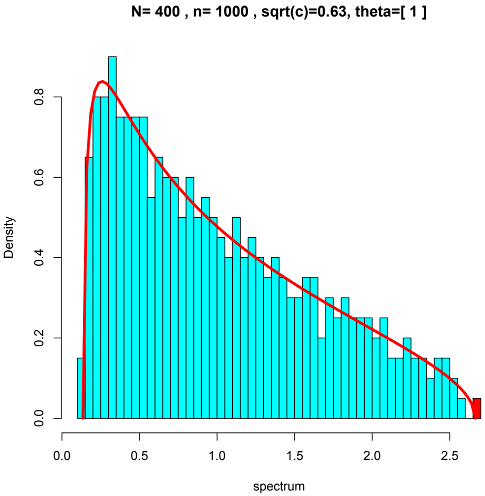

Figure: Spiked model - strength of the perturbation $\theta = 1$

Simulations

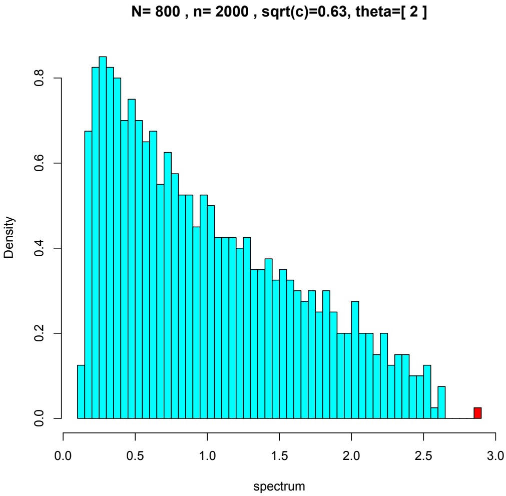

Figure: Spiked model - strength of the perturbation $\theta = 2$

Simulations

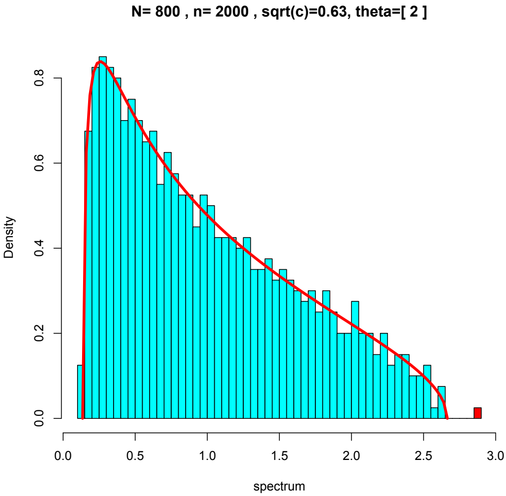

Figure: Spiked model - strength of the perturbation $\theta = 2$

Simulations

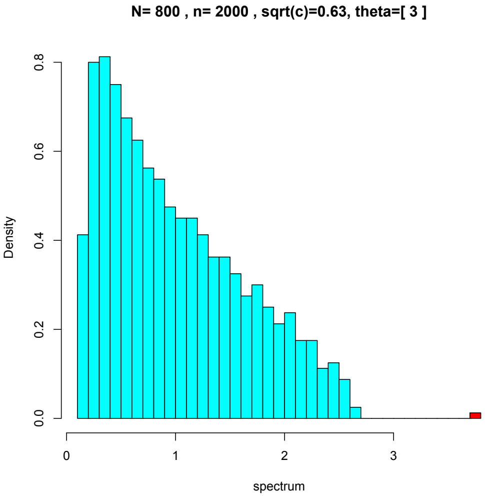

Figure: Spiked model - strength of the perturbation $\theta = 3$

Simulations

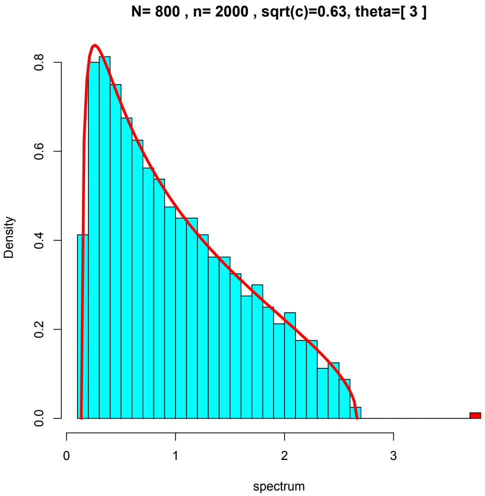

Figure: Spiked model - strength of the perturbation $\theta = 3$

To conclude

Many interesting questions in RMT arise from ecological models.

1. Normalization and feasibility,
2. General interaction matrix models

Network type	**Statistical Features**	**RMT Results**
random	Z_{ij} i.i.d. and $\mathbb{E}Z_{ij} = 0$	Circular law
structured	$Z_{ij} = 0$ for $(i, j) \in \mathcal{S}$	Sparse variance profiles / open
mutualistic	$\mathbb{E}Z_{ij} > 0$	open
predator-pray	$Z_{ij} = -Z_{ji}$	open
density-dependent	$\mathcal{J} = DZ$	open

Figure: Various types of ecological networks

Thank you for your attention !